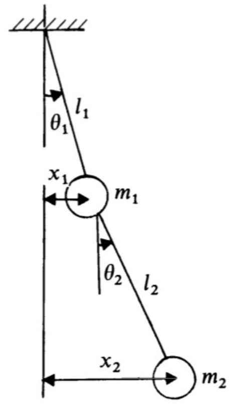
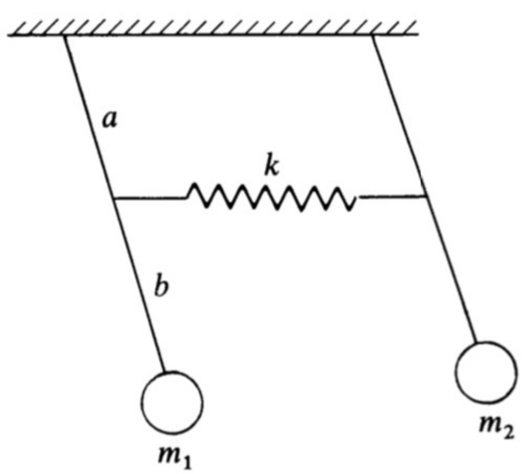
Practice Problem Set 6A
2-DOF: Free Undamped

Do the following problems from the book:

6.3. Determine the differential equations of motion of the double pendulum shown in the figure below in terms of the coordinates x_1 and x_2. Identify the system mass and stiffness matrices.



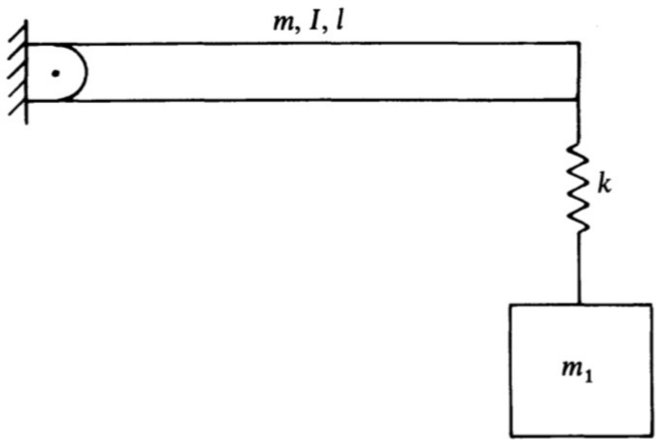
6.7. Write down the differential equations of motion of the two degree of freedom system shown in the figure below. Identify the system mass and stiffness matrices. Obtain the characteristic equation and determine the system natural frequencies and amplitude ratios in the following special case: $m_1 = m_2, \quad a = b$



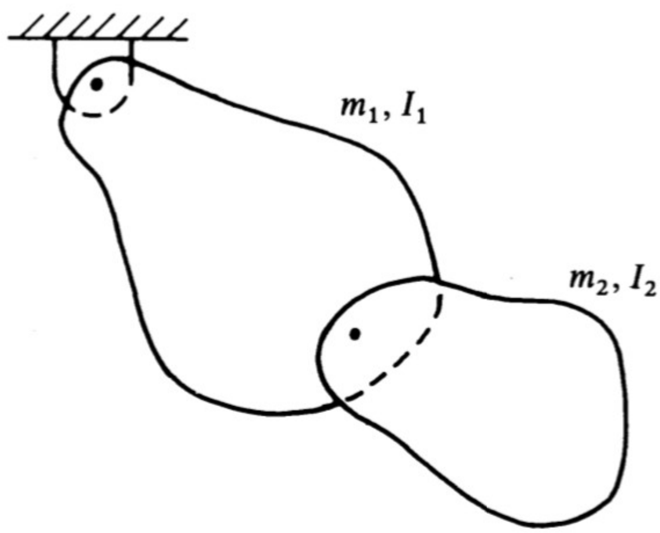
6.8. In Problem 6.7, if $m_1 = m_2$ = 0.5 kg, a = b = 0.25 m, and k = 1000 N/m, determine the response of the system to the initial conditions

$$\theta_1(0)=\theta_2(0)=\dot{\theta}_1(0)=0 \quad \text{and} \quad \dot{\theta}_2(0)=3\,\text{rad/s}$$

6.11. Determine the differential equations of motion of the 2-DOF system shown below. Identify the system mass and stiffness matrices. Obtain the characteristic equation and determine the natural frequencies of the system.



6.12. Determine the differential equations of motion of the 2-DOF system shown below. Identify the system mass and stiffness matrices. Obtain the characteristic equation and determine the natural frequencies ω_1 and ω_2. Assume small oscillations for the two rigid bodies.



SCROLL

DOWN

FOR

SOLUTION

(But don't get tempted by the dark side. Resist! Use the, um, Force?)

ARE

YOU

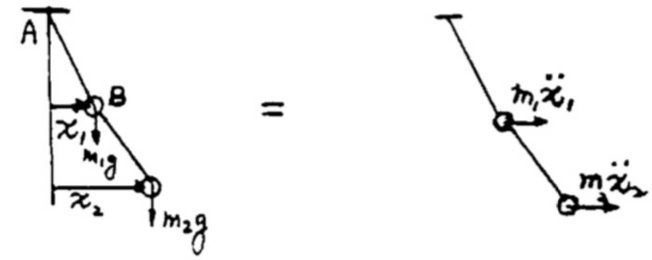
SURE?

(Go back up and think harder? Also, what exactly are you looking for in the solution below?)

SOLUTION

6.3

(3)



$$\Sigma M_A = 0$$

$$m_1 \ddot{x}_1 l_1 + m_2 \ddot{x}_2 (l_1 + l_2) + m_1 g x_1 + m_2 g x_2 = 0$$

$$\Sigma M_B = 0$$

$$m_2 \ddot{x}_2 l_2 + m_2 g (x_2 - x_1) = 0$$

$$\begin{bmatrix} m_1 l_1 & m_2 (l_1 + l_2) \\ 0 & m_2 l_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} m_1 g & m_2 g \\ -m_2 g & m_2 g \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} m_1 l_1 & m_2 l_1 \\ 0 & m_2 l_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} (m_1 + m_2) g & 0 \\ -m_2 g & m_2 g \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{M} = \begin{bmatrix} m_1 l_1 & m_2 l_1 \\ 0 & m_2 l_2 \end{bmatrix}$$

$$\underline{K} = \begin{bmatrix} (m_1 + m_2) g & 0 \\ -m_2 g & m_2 g \end{bmatrix}$$

6.7

(7) $a+b=\ell$

$$m_1\ell^2\ddot{\theta}_1 = -m_1 g\ell\theta_1 + ka^2(\theta_2-\theta_1)$$

$$m_2\ell^2\ddot{\theta}_2 = -m_2 g\ell\theta_2 - ka^2(\theta_2-\theta_1)$$

In matrix form

$$\begin{bmatrix} m_1\ell^2 & 0 \\ 0 & m_2\ell^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} m_1 g\ell + ka^2 & -ka^2 \\ -ka^2 & m_2 g\ell + ka^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $m_1 = m_2 = m$, $a = b = \frac{\ell}{2}$

$$|\underline{K} - \omega^2\underline{M}| = \begin{vmatrix} mg\ell + ka^2 - m\ell^2\omega^2 & -ka^2 \\ -ka^2 & mg\ell + ka^2 - m\ell^2\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow \bar{a}\,\omega^4 + \bar{b}\,\omega^2 + \bar{c} = 0$$

where

$$\bar{a} = m\ell^4$$

$$\bar{b} = -2\ell^3(mg + \tfrac{1}{4}k\ell)$$

$$\bar{c} = \ell^2(mg^2 + \tfrac{1}{2}kg\ell)$$

$$\omega_1^2 = \frac{-\bar{b} + \sqrt{\bar{b}^2 - 4\bar{a}\bar{c}}}{2\bar{a}}$$

$$\omega_2^2 = \frac{-\bar{b} - \sqrt{\bar{b}^2 - 4\bar{a}\bar{c}}}{2\bar{a}}$$

$$\beta_1 = \frac{\theta_{11}}{\theta_{21}} = \frac{\frac{1}{4}k\ell^2}{mg\ell + \frac{1}{4}k\ell^2 - \omega_1^2 m\ell^2} = \frac{mg\ell + \frac{1}{4}k\ell^2 - \omega_1^2 m\ell^2}{\frac{1}{4}k\ell^2}$$

$$\beta_2 = \frac{\theta_{12}}{\theta_{22}} = \frac{\frac{1}{4}k\ell^2}{mg\ell + \frac{1}{4}k\ell^2 - \omega_2^2 m\ell^2} = \frac{mg\ell + \frac{1}{4}k\ell^2 - \omega_2^2 m\ell^2}{\frac{1}{4}k\ell^2}$$

6.8

(8) $m_1 = m_2 = 0.5\,kg$, $a = b = 0.25\,m$

$k = 1000\ N/m$, $\ell = 0.5\,m$

$\bar{a} = m\ell^4 = 0.03125$

$\bar{b} = -2\ell^3(mg + \frac{1}{4}k\ell) = -32.475$

$\bar{c} = \ell^2(mg + \frac{1}{2}kg\ell) = 613.725$

$\omega_1^2 = 1019.945$, $\omega_2^2 = 19.255$

$\omega_1 = 31.94$ rad/s , $\omega_2 = 4.39$ rad/s

$$\beta_1 = \frac{\theta_{11}}{\theta_{21}} = \frac{k\ell^2}{4mg\ell + k\ell^2 - 4\omega_1^2 m\ell^2} = -1$$

$$\beta_2 = \frac{\theta_{12}}{\theta_{22}} = \frac{k\ell^2}{4mg\ell + k\ell^2 - 4\omega_2^2 m\ell^2} = 1$$

I.C. $\begin{cases} \theta_{10} = 0 , & \dot{\theta}_{10} = 0 \\ \theta_{20} = 0 , & \dot{\theta}_{20} = 3 \end{cases}$

$0 = -\theta_{21}\sin\phi_1 + \theta_{22}\sin\phi_2$

$0 = \theta_{21}\sin\phi_1 + \theta_{22}\sin\phi_2$

$0 = -31.94\,\theta_{21}\cos\phi_1 + 4.39\cos\phi_2$

$3 = 31.94\,\theta_{21}\cos\phi_1 + 4.39\cos\phi_2$

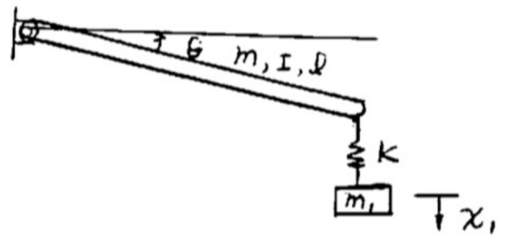
$\begin{cases} \theta_{21} = 0.047 , & \phi_1 = 0 \\ \theta_{22} = 0.342 , & \phi_2 = 0 \end{cases}$

$\therefore\ \theta_1(t) = -0.047\sin(31.94t) + 0.342\sin(4.39t)$

$\theta_2(t) = 0.047\sin(31.94t) + 0.342\sin(4.39t)$

6.11

(11)



Eq. of motion

$$\left(I+\frac{ml^2}{4}\right)\ddot{\theta} = kl(x_1 - l\theta)$$

$$m_1\ddot{x}_1 = -k(x_1 - l\theta)$$

In matrix form

$$\begin{bmatrix} I+\frac{ml^2}{4} & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} kl^2 & -kl \\ -kl & k \end{bmatrix}\begin{bmatrix} \theta \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Characteristic Eq.

$$\left|\underline{k} - \omega^2\underline{M}\right| = \begin{vmatrix} kl^2 - \omega^2\left(I+\frac{ml^2}{4}\right) & -kl \\ -kl & k-\omega^2 m_1 \end{vmatrix} = 0$$

$$\Rightarrow \quad a\omega^4 + b\omega^2 + c = 0$$

where $a = \left(I + \frac{ml^2}{4}\right)m_1$

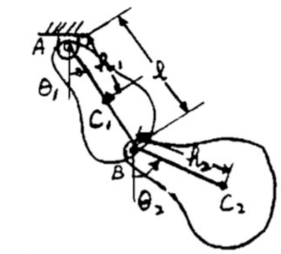
$b = -\left[k\left(I+\frac{ml^2}{4}\right) + kl^2 m_1\right]$

$c = 0$

$$\omega_1^2 = 0 \quad , \quad \omega_2^2 = -\frac{b}{a}$$

6.12

(12)



$\Sigma M_A = 0$

$$I_1 \ddot{\theta}_1 + I_2 \ddot{\theta}_2 + m_1 h_1^2 \ddot{\theta}_1 + m_2 (l \ddot{\theta}_1 + h_2 \ddot{\theta}_2)(l + h_2)$$

$$+ m_1 g h_1 \theta_1 + m_2 g (l \theta_1 + h_2 \theta_2) = 0$$

$\Sigma M_B = 0$

$$I_2 \ddot{\theta}_2 + m_2 (l \ddot{\theta}_1 + h_2 \ddot{\theta}_2) h_2 + m_2 g h_2 \theta_2 = 0$$

In matrix form

$$\begin{bmatrix} I_1 + m_1 h_1^2 + m_2 l(l + h_2) & I_2 + m_2 h_2 (l + h_2) \\ m_2 l h_2 & I_2 + m_2 h_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} (m_1 h_1 + m_2 l) g & m_2 h_2 g \\ 0 & m_2 h_2 g \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} I_1 + m_1 h_1^2 + m_2 l^2 & m_2 l h \\ m_2 l h & I + m_2 h_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} (m_1 h_1 + m_2 l) g & 0 \\ 0 & m_2 h_2 g \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Characteristic Eq.

$$| \underline{k} - \omega^2 \underline{M} | = 0 \Rightarrow a\omega^4 + b\omega^2 + c = 0$$

where $a = (I_1 + m_1 h_1^2)(I_2 + m_2 h_2^2)$

$$b = -[m_1 g h_1 (I_2 + m_2 h_2^2) + m_2 g h_2 (I_1 + m_1 h_1^2)]$$

$$c = 0$$

$$\omega_1^2 = 0 \Rightarrow \omega_1 = 0$$

$$\omega_2^2 = -\frac{b}{a} \Rightarrow \omega_2 = \sqrt{-\frac{b}{a}}$$