Practice Problem Set 5D
Arbitrary Forced Vibration

Do the following problems from the book:

5.23. If an arbitrary force $F(t)$ is applied to an undamped single degree of freedom mass–spring system with nonzero initial conditions, show that the response of the system must be written in the form

$$x(t) = x_0 \cos \omega t + \frac{\dot{x}_0}{\omega} \sin \omega t + \frac{1}{m\omega} \int_0^t F(\tau) \sin \omega(t - \tau)\, d\tau$$

where x_0 is the initial displacement and $\dot{x}_0$ is the initial velocity.

5.24. Determine the forced response of the undamped single degree of freedom spring–mass system to the forcing function shown in Fig. P11.

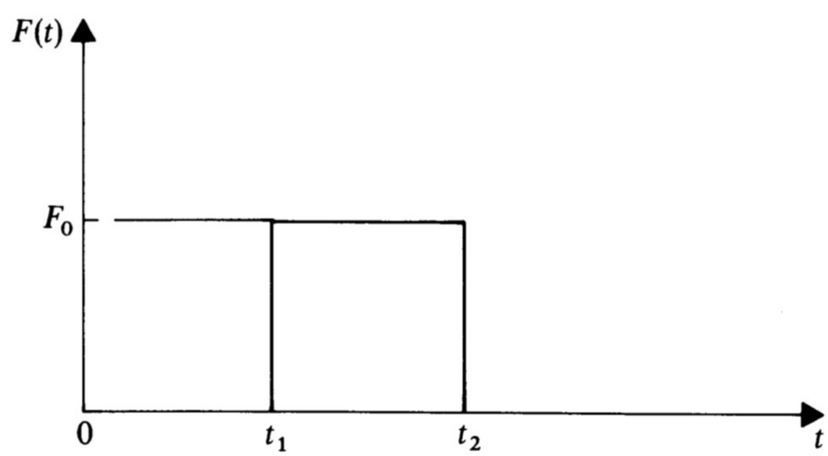

FIG. P5.11

SCROLL

DOWN

FOR

SOLUTION

(But don't get tempted by the dark side. Resist! Use the, um, Force?)

ARE

YOU

SURE?

(Go back up and think harder? Also, what exactly are you looking for in the solution below?)

SOLUTION

5.23 (Note: Eq. 5.50 is the solution equation involving Duhamel integral, at timestamp 3:03 of the lecture video on Youtube)

(23)

$$m\ddot{x} + kx = F$$

$$\omega = \sqrt{\frac{k}{m}}, \quad \zeta = 0$$

$$x_h = A_1 \cos\omega t + A_2 \sin\omega t$$

For arbitrary force function,

Eq. (5.50) can be reduced ($\zeta = 0, \omega_d = \omega$)

$$x_p = \frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau$$

$$\therefore x(t) = x_h + x_p$$

$$= A_1 \cos\omega t + A_2 \sin\omega t + \frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau$$

I.C.

$$\begin{cases} x(0) = x_0 & \Rightarrow \quad A_1 = x_0 \\ \dot{x}(0) = \dot{x}_0 & \Rightarrow \quad A_2 = \dfrac{\dot{x}_0}{\omega} \end{cases}$$

Therefore

$$x(t) = x_0 \cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t + \frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau$$

5.24

(24) From Prob. 23, we have

$$x(t) = x_0 \cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t + \frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau$$

$$F(\tau) = \begin{cases} F_0 & t_1 \le \tau \le t_2 \\ 0 & \text{else} \end{cases}$$

$$\frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau$$

$$= \frac{1}{m\omega}\int_{t_1}^{t_2} F_0 \sin\omega(t-\tau)\,d\tau$$

$$= \frac{F_0}{m\omega^2}[\cos\omega(t-t_2) - \cos\omega(t-t_1)]$$

$$= \frac{2F_0}{m\omega^2}\sin\omega\left(\frac{t_2-t_1}{2}\right)\sin\omega\left[t-\frac{(t_1+t_2)}{2}\right]$$

$$\therefore\ x(t) = x_0\cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t + \frac{2F_0}{m\omega^2}\sin\omega\left(\frac{t_2-t_1}{2}\right)\sin\omega\left[t-\frac{(t_1+t_2)}{2}\right]$$